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Report

Kinematics of a positioning table with three vertical jacks

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Summary

This report shows the derivation of the kinematic equations for a positioning table which is actuated by three vertical jacks.

Contents

1 Preface

At several beamline-end-stations a positioning table as shown in Figure 1 is used to put the sample into the beam. Since the position and orientation of the sample with respect to the beam are of high interest it is necessary to know the kinematic behavior of the sample with respect to the three vertical actuators. This is what the following report deals with.

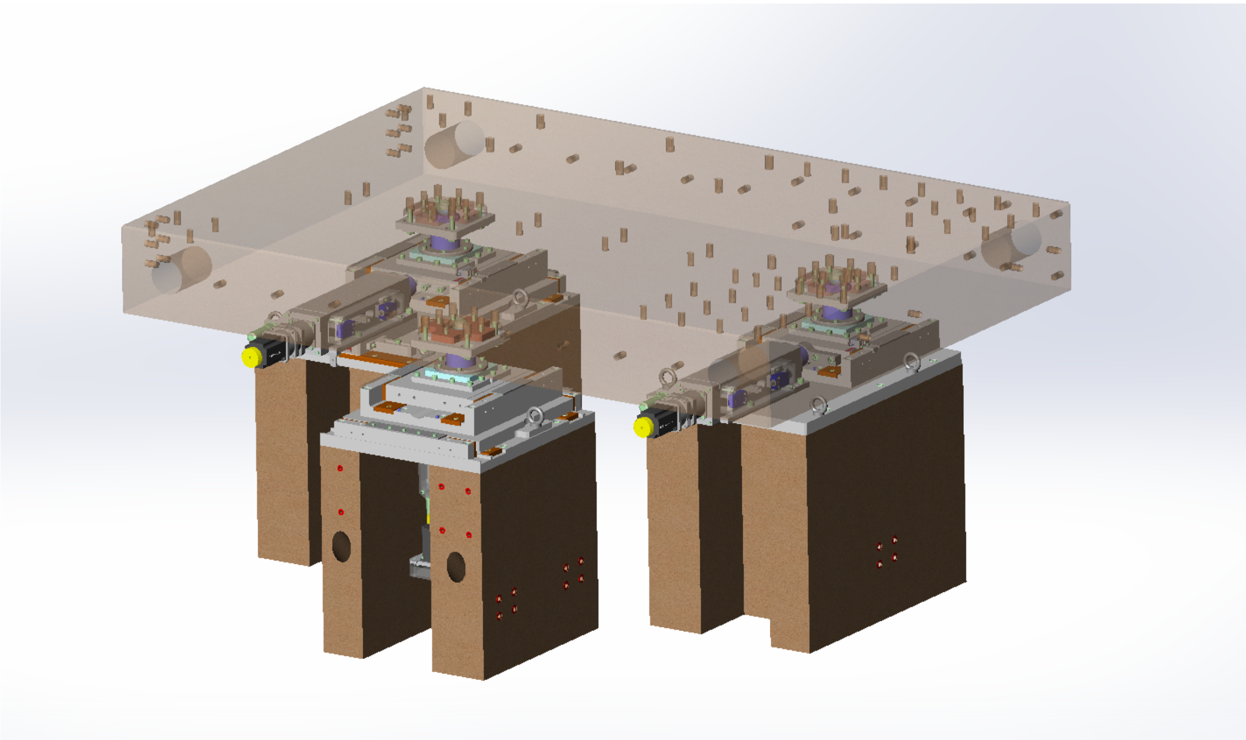

Figure 1: COD model of a positioning table

2 Modeling of the system

The modeling of the system has been done as shown in Figure 2. The world frame $\{w\}$ gives the orientation of the beam and is used to define the positions (a, b, c) of the actuators. On the other hand the sample frame $\{s\}$ is used to give the position and orientation of the sample with respect to the three actuators.

For computational issues there will be also a third coordinate frame introduced in Figure 5 which has the three coordinates e_1, e_2 and e_3 instead of x, y and z. It is used to simplify the computation and it is defined such that the e_1 coordinate shows always in the free direction of the actuator which can either move in the x or in the y direction.

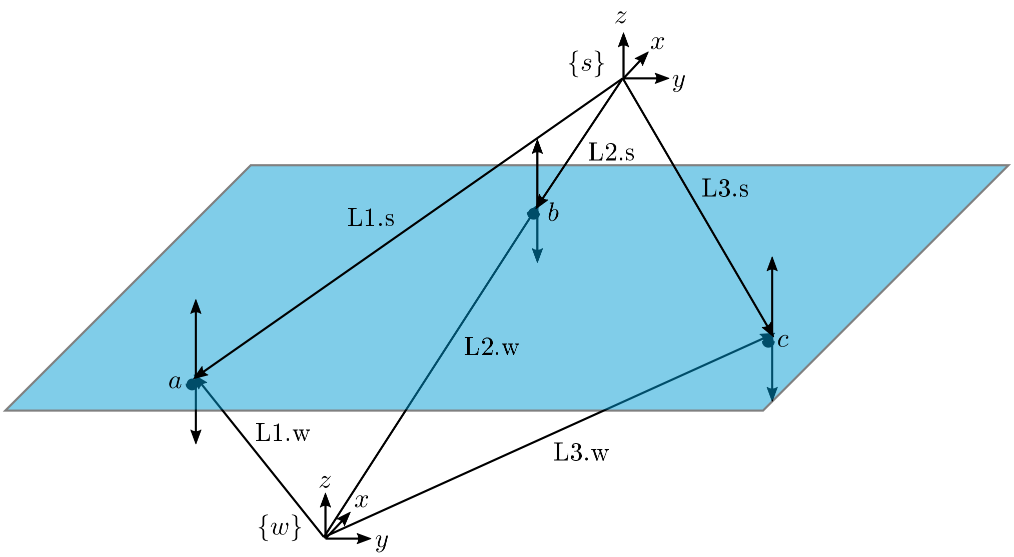

Figure 2: Kinematic model of the positioning table

3 Inverse kinematic

The inverse kinematic is the computation of the actuator positions according to a desired position of the sample. Since there are only three actuators used for the positioning of the sample table, only three dof (degrees of freedom) can be adjusted arbitrary. These dof are the position in z direction and the rotations about the x and the y axis. The other three dof are constrained by the geometry of the setup. For the inverse kinematic it is hence, necessary to define first the desired values to be z, r_x and r_y. This gives the following rotation matrix for the sample frame

$$\mathbf{R}(r_x, r_y) = \begin{bmatrix} \cos(r_y) & 0 & \sin(r_y) \\ 0 & 1 & 0 \\ -\sin(r_y) & 0 & \cos(r_y) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(r_x) & -\sin(r_x) \\ 0 & \sin(r_x) & \cos(rx) \end{bmatrix} = \begin{bmatrix} c_{r_y} & s_{r_y}s_{r_x} & s_{r_y}c_{r_x} \\ 0 & c_{r_x} & -s_{r_x} \\ -s_{r_y} & c_{r_y}s_{r_x} & c_{r_y}c_{r_x} \end{bmatrix} \tag{1}$$

In the last matrix the acronyms s_x and c_x have been used for $\sin(x)$ and $\cos(x)$. Given this rotation matrix it is now easy to compute the needed positions of the actuators to get the table into the desired orientation. Since one of the actuators is fixed in the x-y plane this actor can be used for defining a kinematic chain, see Figure 3.

This kinematic chain can be written with homogeneous transformation matrices as

$$\begin{bmatrix} \mathbf{I}_3 & -L1.s \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(r_x, r_y) & p \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}(r_x, r_y) & \underline{0} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & -L1.s \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{2}$$

where $\underline{0} = [0 \quad 0 \quad 0]^{\mathrm{T}}$ and $\mathbf{I}_3$ is the identity matrix of order 3. But since the motion of the sample frame is not completely free, a constrain equation can be found, which defines the parasitic rotation about the z-axis. Therefore a kinematic chain containing the actuator which is fixed and the one which has only one dof in the x-y plane. This chain can be seen in Figure 4 and the equation (in homogeneous transformation matrices) boils down to

$$\begin{bmatrix} \mathbf{I}_3 & p_{bb'} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & p_{ba} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(r_x, r_y) & \underline{0} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(r_z) & \underline{0} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & p_{ab'} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{3}$$

Now it has to be distinguished if the second actuator can move in x or in y direction. According to this dof either the first or the second element in the vector $p_{bb'}$ on the left hand side of equation (3) has to be equal to 0, and hence the angle r_z is defined by this.

With the known parasitic rotation about the z axis the orientation and of the sample is completely

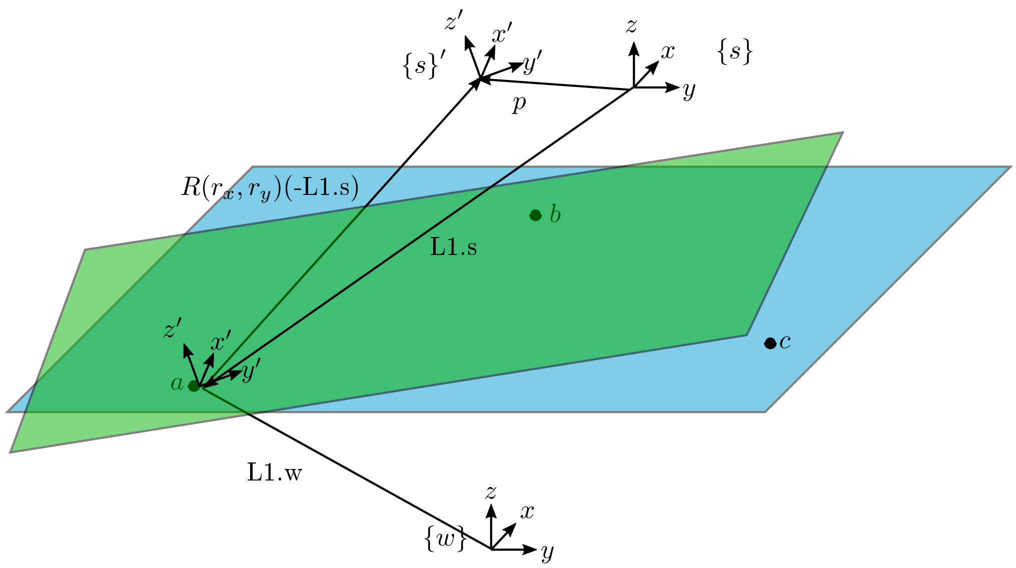

Figure 3: Kinematic chain for the actuator fixed in the x-y plane

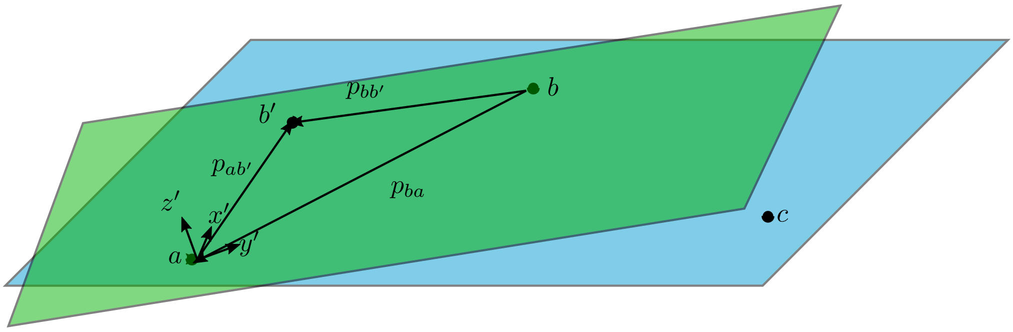

Figure 4: Kinematic chain for the constrained motion

defined. So equation (3) can be used to find the needed position $p_{bb'}$ of the actor b. The same kinematic chain can now be defined for the third actuator in order to find the third position $p_{cc'}$.

$$\begin{bmatrix} \mathbf{I}_3 & p_{cc'} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & p_{ca} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(r_x, r_y) & \underline{0} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(r_z) & \underline{0} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & p_{ac'} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{4}$$

From the both vectors $p_{bb'}$ and $p_{cc'}$ the last element (the one in z direction) gives the actuation value of the actor to reach the desired orientation. As last parameter now the height of the sample frame is still adjustable, so that the desired value is going to be reached. Therefore it is the easiest to rewrite equation (2), so that the vector p can be computed

$$\begin{bmatrix} \mathbf{R}(r_x, r_y) & p \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & L1.s \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(r_x, r_y) & \underline{0} \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & -L1.s \\ \underline{0}^{\mathrm{T}} & 1 \end{bmatrix} \quad . \tag{5}$$

The difference from the so calculated z position (third element of p) and the desired z position now has to be added to the values found by the equations for the orientation.

$$\Delta z = z_{in} - p(3) \tag{6}$$
$$z_a = \Delta z \tag{7}$$
$$z_b = \Delta z + p_{bb'}(3) \tag{8}$$
$$z_c = \Delta z + p_{cc'}(3) \tag{9}$$

Now equations (7) to (9) give the needed actuation positions.

4 Forward kinematics

In order to compute the position and orientation of the sample by a given set of actuator positions the so called forward kinematic is needed. Therefore it is first needed to compute the positions of the three jacks in the x-y plain. This can be done by using the constrains of the euclidean distance of the three jacks, as shown in Figure 5. This give the following three equations.

$$\|p_{ab}\|_2 \stackrel{!}{=} \|p_{ab'}\|_2 = \|p_{ab} + p_{bb'}\|_2 \tag{10}$$
$$\|p_{ac}\|_2 \stackrel{!}{=} \|p_{ac'}\|_2 = \|p_{ac} + p_{cc'}\|_2 \tag{11}$$
$$\|p_{bc}\|_2 \stackrel{!}{=} \|p_{b'c'}\|_2 = \| - p_{bb'} + p_{bc} + p_{cc'}\|_2 \tag{12}$$

This gives three equations for three variables (x and y coordinate of $p_{cc'}$ and x or y coordinate of $p_{bb'}$). Since the equations are nonlinear, an analytic solution is quite complex but the Newton method can be used for finding the solution numerically.

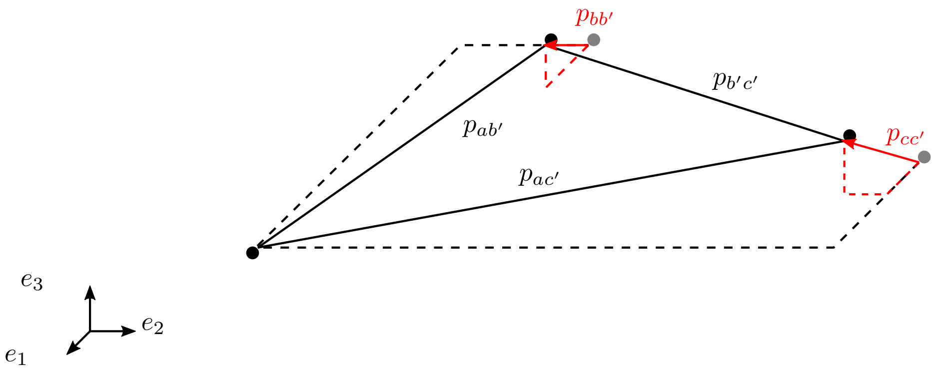

Figure 5: Vector definitions for the forward kinematics

After $p_{bb'}$ and $p_{cc'}$ are found, we can define

$$p_{ab'} = p_{ab} + p_{bb'} \tag{13}$$
$$p_{ac'} = p_{ac} + p_{cc'} \tag{14}$$
$$p_{b'c'} = -p_{bb'} + p_{bc} + p_{cc'} \tag{15}$$

and with the rotation matrix $\mathbf{R}(r_x, r_y, r_z)$ follows

$$p_{ab'} = \mathbf{R}(r_x, r_y, r_z)p_{ab} \tag{16}$$
$$p_{ac'} = \mathbf{R}(r_x, r_y, r_z)p_{ac} \tag{17}$$
$$p_{b'c'} = \mathbf{R}(r_x, r_y, r_z)p_{bc} \tag{18}$$

$$\mathbf{R}(r_x, r_y, r_z) = \mathbf{R}(r_x, r_y)\mathbf{R}(r_z) = \begin{bmatrix} c_{r_y}c_{r_z} + s_{r_x}s_{r_y}s_{r_z} & s_{r_y}s_{r_x}c_{r_z} - c_{r_y}s_{r_z} & s_{r_y}c_{r_x} \\ c_{r_x}s_{r_z} & c_{r_x}c_{r_z} & -s_{r_x} \\ c_{r_y}s_{r_x}s_{r_z} - c_{r_z}s_{r_y} & s_{r_y}s_{r_z} + c_{r_y}c_{r_z}s_{r_x} & c_{r_y}c_{r_x} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{19}$$

The equations (16) to (18) can now be rearranged to find the rotation matrix

$$\begin{bmatrix} p_{ab'} \\ p_{ac'} \\ p_{b'c'} \end{bmatrix} = \begin{bmatrix} p_{ab}(1)\mathbf{I}_3 & p_{ab}(2)\mathbf{I}_3 & p_{ab}(3)\mathbf{I}_3 \\ p_{ac}(1)\mathbf{I}_3 & p_{ac}(2)\mathbf{I}_3 & p_{ac}(3)\mathbf{I}_3 \\ p_{bc}(1)\mathbf{I}_3 & p_{bc}(2)\mathbf{I}_3 & p_{bc}(3)\mathbf{I}_3 \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \\ r_{12} \\ r_{22} \\ r_{32} \\ r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \tag{20}$$

Inversion of equation (20) gives the entries of the Rotation matrix. From these entries the rotation angels can be computed as followed:

$$\frac{r_{21}}{r_{22}} = \frac{c_{r_x} s_{r_z}}{c_{r_x} c_{r_z}} = \frac{s_{r_z}}{c_{r_z}} = \tan(r_z) \qquad \Rightarrow \qquad r_z = \operatorname{atan}\left(\frac{r_{21}}{r_{22}}\right) \tag{21}$$

$$r_{22} = c_{r_x} c_{r_z} \qquad \Rightarrow \qquad r_x = \operatorname{acos}\left(\frac{r_{22}}{\cos(r_z)}\right) \tag{22}$$

$$\begin{bmatrix} r_{11} \\ r_{31} \end{bmatrix} = \begin{bmatrix} c_{r_z} & s_{r_x} s_{r_z} \\ s_{r_x} s_{r_z} & -c_{r_z} \end{bmatrix} \begin{bmatrix} c_{r_y} \\ s_{r_y} \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} c_{r_y} \\ s_{r_y} \end{bmatrix} = \begin{bmatrix} c_{r_z} & s_{r_x} s_{r_z} \\ s_{r_x} s_{r_z} & -c_{r_z} \end{bmatrix}^{-1} \begin{bmatrix} r_{11} \\ r_{31} \end{bmatrix}$$

$$r_y = \operatorname{atan}\left(\frac{s_{r_y}}{c_{r_y}}\right) \tag{23}$$

The elements r_{13}, r_{23} and r_{33} have not been used for the computation of r_x, r_y and r_z, since the third entry of the vectors p_{ab}, p_{ac} and p_{bc} can be 0 and in this case equation (20) becomes

$$\begin{bmatrix} p_{ab'} \\ p_{ac'} \end{bmatrix} = \begin{bmatrix} p_{ab}(1)\mathbf{I}_3 & p_{ab}(2)\mathbf{I}_3 \\ p_{ac}(1)\mathbf{I}_3 & p_{ac}(2)\mathbf{I}_3 \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \\ r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} . \tag{24}$$